W47. Let $x,y,z > 0$ such that $(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = \frac{91}{10}$

Compute $\left\lfloor (x^3+y^3+z^3)\left(\frac{1}{x^3}+\frac{1}{y^3}+\frac{1}{z^3}\right)\right\rfloor$

where $\lfloor \cdot \rfloor$ represent the integer part.

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Solution by Arkady Alt , San Jose ,California, USA.

Let $a := \frac{y}{z}+\frac{z}{y}, b := \frac{z}{x}+\frac{x}{z}, c := \frac{x}{y}+\frac{y}{z}$. Then $a,b,c \geq 2$ and

$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = \frac{91}{10} \Leftrightarrow \sum_{cyc}\left(\frac{x}{y}+\frac{y}{x}\right)+3 = \frac{91}{10} \Leftrightarrow$

$a+b+c = \frac{61}{10}$. Also we have $(x^3+y^3+z^3)\left(\frac{1}{x^3}+\frac{1}{y^3}+\frac{1}{z^3}\right) =$

$\sum_{cyc}\left(\frac{x^3}{y^3}+\frac{y^3}{x^3}\right)+3 = \sum_{cyc}\left(\left(\frac{x}{y}+\frac{y}{x}\right)^3-3\left(\frac{x}{y}+\frac{y}{x}\right)\right)+3 =$

$\sum_{cyc}\left(\frac{x}{y}+\frac{y}{x}\right)^3-3\sum_{cyc}\left(\frac{x}{y}+\frac{y}{x}\right)+3 = a^3+b^3+c^3-3(a+b+c)+3 =$

$a^3+b^3+c^3-3\cdot\frac{61}{10}+3 = a^3+b^3+c^3-\frac{153}{10}$.

Consider now the following problem:

For any real $k > 6$ find $\max\left\{a^3+b^3+c^3 \mid a,b,c \geq 2 \text{ and } a+b+c = k\right\}$.

(in our case $k = \frac{61}{10}$). Due to symmetry we may assume that $a \leq b \leq c$.

Since $\begin{cases} a+b+c = k \\ a,b,c \geq 2 \end{cases} \Leftrightarrow \begin{cases} \frac{k}{3} \leq c \leq k-4 \\ 2 \leq a \leq \frac{k-c}{2} \\ b = k-c-a \end{cases}$ and $a^3+b^3+c^3 =$

$a^3+(k-c-a)^3+c^3 = 3(k-c)(a^2-(k-c)a)+k(k^2-3ck+3c^2)$,

where quadratic function $a^2-(k-c)a$ decrease in $\left[2,\frac{k-c}{2}\right]$, then

$$3(k-c)(a^2-(k-c)a)+k(k^2-3ck+3c^2) \leq$$
$$3(k-c)(2^2-(k-c)\cdot 2)+k(k^2-3ck+3c^2) =$$
$$3(k-2)(c^2-(k-2)c)+k(k^2-6k+12).$$

Since $\frac{k-2}{2} \in \left[\frac{k}{3},k-4\right]$ then $\max\left\{c^2-(k-2)c \mid \frac{k}{3} \leq c \leq k-4\right\} =$

$\max\left\{\left(\frac{k}{3}\right)^2-(k-2)\cdot\frac{k}{3},(k-4)^2-(k-2)(k-4)\right\} =$

$\max\left\{\frac{2}{3}k-\frac{2}{9}k^2,8-2k\right\} = 8-2k$ and, therefore,

$3(k-2)(c^2-(k-2)c)+k(k^2-6k+12) \leq 3(k-2)(8-2k)+k(k^2-6k+12) =$

$k^3-12k^2+48k-48$.

Thus, $a^3+b^3+c^3 \leq k^3-12k^2+48k-48$ and in particular for $k = \frac{61}{10}$ we obtain

$a^3+b^3+c^3 \leq \left(\frac{61}{10}\right)^3-12\left(\frac{61}{10}\right)^2+48\left(\frac{61}{10}\right)-48 = \frac{25261}{1000} = 25.261$.

Coming back to original notations we get

$$(x^3+y^3+z^3)\left(\frac{1}{x^3}+\frac{1}{y^3}+\frac{1}{z^3}\right) \le 25.261 - 15.3 = 9.961.$$

Since by Cauchy Inequality $(x^3+y^3+z^3)\left(\frac{1}{x^3}+\frac{1}{y^3}+\frac{1}{z^3}\right) \ge 9$

then $\left\lfloor (x^3+y^3+z^3)\left(\frac{1}{x^3}+\frac{1}{y^3}+\frac{1}{z^3}\right) \right\rfloor = 9$